

SYZYGY BUNDLES ON \mathbb{P}^2 AND THE WEAK LEFSCHETZ PROPERTY

HOLGER BRENNER AND ALMAR KAID

ABSTRACT. Let K be an algebraically closed field of characteristic zero and let $I = (f_1, \dots, f_n)$ be a homogeneous R_+ -primary ideal in $R := K[X, Y, Z]$. If the corresponding syzygy bundle $\text{Syz}(f_1, \dots, f_n)$ on the projective plane is semistable, we show that the Artinian algebra R/I has the Weak Lefschetz property if and only if the syzygy bundle has a special generic splitting type. As a corollary we get the result of Harima et al., that every Artinian complete intersection ($n = 3$) has the Weak Lefschetz property. Furthermore, we show that an almost complete intersection ($n = 4$) does not necessarily have the Weak Lefschetz property, answering negatively a question of Migliore and Miró-Roig. We prove that an almost complete intersection has the Weak Lefschetz property if the corresponding syzygy bundle is not semistable.

Mathematical Subject Classification (2000): primary: 13D02, 14J60, secondary: 13C13, 13C40, 14F05.

Keywords: syzygy, semistable bundle, Grauert-Mülich Theorem, Weak Lefschetz property, Artinian algebra, complete intersection, almost complete intersection.

1. INTRODUCTION

Throughout this paper we denote by $R := K[X_0, \dots, X_N]$ a polynomial ring in $N + 1$ variables over an algebraically closed field K of characteristic zero. A family of R_+ -primary homogeneous polynomials f_1, \dots, f_n of degree d_i in R defines on $\mathbb{P}^N = \mathbb{P}_K^N = \text{Proj } R$ the short exact sequence

$$0 \longrightarrow \text{Syz}(f_1, \dots, f_n)(m) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(m - d_i) \xrightarrow{f_1, \dots, f_n} \mathcal{O}_{\mathbb{P}^N}(m) \longrightarrow 0$$

of locally free sheaves. We call the vector bundle of rank $n - 1$ on the left the *syzygy bundle* of the elements f_1, \dots, f_n . On the other hand the elements f_1, \dots, f_n define an Artinian graded R -algebra $A := R/(f_1, \dots, f_n)$, i.e. A is of the form

$$A = K \oplus A_1 \oplus \dots \oplus A_s$$

for an integer $s \geq 0$. The algebra A has the so called *Weak Lefschetz property* (*WLP*) if for every general linear form $\ell \in R_1$ the multiplication maps

$$A_m \xrightarrow{\cdot \ell} A_{m+1}$$

have maximal rank for $m = 0, \dots, s-1$, i.e. these K -linear maps are either injective or surjective.

In case of generic elements $f_1, \dots, f_n \in R$, the Weak Lefschetz property is related to the study of the Fröberg conjecture. This conjecture is equivalent to the *Maximal Rank property*, i.e. the property that the multiplication maps have maximal rank for every $d \geq 1$ and every general form $F \in R_d$. The Fröberg conjecture is only known for the cases $N = 1, 2$ (cf. [3] and [1]).

Our objective is to study for $N = 2$ the connection between the WLP for the algebra A and the semistability of the syzygy bundle $\text{Syz}(f_1, \dots, f_n)$. We recall that a torsion-free coherent sheaf \mathcal{E} on \mathbb{P}^N is *semistable* (in the sense of Mumford and Takemoto) if for every coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ the inequality $\frac{\deg(\mathcal{F})}{\text{rk}(\mathcal{F})} \leq \frac{\deg(\mathcal{E})}{\text{rk}(\mathcal{E})}$ holds (and *stable* if $<$ holds), where the *degree* $\deg(\mathcal{F})$ is defined as the twist k such that $(\bigwedge^{\text{rk}(\mathcal{F})} \mathcal{F})^{**} \cong \mathcal{O}_{\mathbb{P}^N}(k)$ (cf. [7] and [9]). By the Theorem of *Grauert-Mülich* (cf. [7, Theorem 3.0.1] or [9, Corollary 1 of Theorem 2.1.4]) a semistable vector bundle \mathcal{E} of rank r on \mathbb{P}^N splits on a generic line $L \subset \mathbb{P}^N$ as $\mathcal{E}|_L \cong \mathcal{O}_L(a_1) \oplus \dots \oplus \mathcal{O}_L(a_r)$ with $a_1 \geq \dots \geq a_r$ and $0 \leq a_i - a_{i+1} \leq 1$ for $i = 1, \dots, r-1$. We prove in Theorem 2.2 that in case of a semistable syzygy bundle on \mathbb{P}^2 the algebra A has *WLP* if and only if in the generic splitting type of $\text{Syz}(f_1, \dots, f_n)$ at most two different twists occur. The importance of the Theorem of Grauert-Mülich was already mentioned in [4], but only for complete intersections in \mathbb{P}^2 .

In the case of an almost complete intersection (i.e. four ideal generators) we show in Example 3.1 that for a semistable syzygy bundle all numerically possible splitting types do actually exist. It follows that there are examples of almost complete intersections where the WLP does not hold, which gives a negative answer to a question of Migliore and Miró-Roig (cf. [8, Paragraph after Question 4.2]). Furthermore, we prove in Theorem 3.3 that in the non semistable case an almost complete intersection has always the Weak Lefschetz property.

2. GENERIC SPLITTING TYPE OF SYZYGY BUNDLES AND THE WEAK LEFSCHETZ PROPERTY

We start with the following cohomological observation.

Proposition 2.1. *Let $R = K[X_0, \dots, X_N]$, $N \geq 2$, and let $I = (f_1, \dots, f_n) \subseteq R$ be an R_+ -primary homogeneous ideal. Then we have*

$$A_m = H^1(\mathbb{P}^N, \text{Syz}(f_1, \dots, f_n)(m))$$

for every graded component A_m of $A := R/I$, $m \in \mathbb{Z}$.

Proof. Since $H^1(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) = 0$ for all m and $N \geq 2$, we derive from the presenting sequence of $\mathcal{S}(m) := \text{Syz}(f_1, \dots, f_n)(m)$ the exact cohomology sequence

$$\bigoplus_{i=1}^n H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m - d_i)) \xrightarrow{f_1, \dots, f_n} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \longrightarrow H^1(\mathbb{P}^N, \mathcal{S}(m)) \longrightarrow 0.$$

Now the claim follows immediately, since $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) = R_m$. \square

We restrict now to three variables and write $R = K[X, Y, Z]$. As usual let f_1, \dots, f_n be homogeneous R_+ -primary elements and let $0 \neq \ell \in R_1$ be a linear form. Thus ℓ defines an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\cdot \ell} \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow \mathcal{O}_L(1) \longrightarrow 0,$$

where the map on the right is the restriction to the line $L = V_+(\ell) \subset \mathbb{P}^2$ defined by ℓ . Since the syzygy bundle $\mathcal{S}(m) := \text{Syz}(f_1, \dots, f_n)(m)$ is locally free this yields the short exact sequence

$$0 \longrightarrow \mathcal{S}(m) \xrightarrow{\cdot \ell} \mathcal{S}(m+1) \longrightarrow \mathcal{S}_L(m+1) \longrightarrow 0$$

(where $\mathcal{S}_L := \mathcal{S}|_L$), and from this we derive the long exact sequence in cohomology

$$\begin{aligned} 0 &\longrightarrow H^0(\mathbb{P}^2, \mathcal{S}(m)) \xrightarrow{\cdot \ell} H^0(\mathbb{P}^2, \mathcal{S}(m+1)) \longrightarrow H^0(L, \mathcal{S}_L(m+1)) \\ &\xrightarrow{\delta} A_m = H^1(\mathbb{P}^2, \mathcal{S}(m)) \xrightarrow{\cdot \ell} A_{m+1} = H^1(\mathbb{P}^2, \mathcal{S}(m+1)) \\ &\longrightarrow H^1(L, \mathcal{S}_L(m+1)) \xrightarrow{\delta} H^2(\mathbb{P}^2, \mathcal{S}(m)). \end{aligned}$$

Hence the Artinian algebra $A = R/(f_1, \dots, f_n)$ has the Weak Lefschetz property if and only if for every generic line $L \subset \mathbb{P}^2$ the map $H^1(\mathbb{P}^2, \mathcal{S}(m)) \rightarrow H^1(\mathbb{P}^2, \mathcal{S}(m+1))$ is either injective or surjective. The injectivity is equivalent to the surjectivity of the restriction map $H^0(\mathbb{P}^2, \mathcal{S}(m+1)) \rightarrow H^0(L, \mathcal{S}_L(m+1))$ and holds in particular when $H^0(L, \mathcal{S}_L(m+1)) = 0$. The surjectivity is equivalent to the injectivity of $H^1(L, \mathcal{S}_L(m+1)) \xrightarrow{\delta} H^2(\mathbb{P}^2, \mathcal{S}(m))$ and holds in particular when $H^1(L, \mathcal{S}_L(m+1)) = 0$. From now on we denote the map $A_m \xrightarrow{\cdot \ell} A_{m+1}$ by μ_ℓ .

The following theorem relates the generic splitting type of a semistable syzygy bundle with the Weak Lefschetz property of the Artinian algebra A .

Theorem 2.2. *Let f_1, \dots, f_n be R_+ -primary homogeneous polynomials in $R = K[X, Y, Z]$ such that their syzygy bundle $\mathcal{S} := \text{Syz}(f_1, \dots, f_n)$ is semistable on \mathbb{P}^2 . Then the following holds:*

- (1) *If the restriction of \mathcal{S} splits on a generic line L as*

$$\mathcal{S}|_L \cong \bigoplus_{i=1}^s \mathcal{O}_L(a+1) \oplus \bigoplus_{i=s+1}^{n-1} \mathcal{O}_L(a),$$

then $A = R/(f_1, \dots, f_n)$ has WLP.

(2) If the restriction of \mathcal{S} splits on a generic line L as

$$\mathcal{S}|_L \cong \mathcal{O}_L(a_1) \oplus \dots \oplus \mathcal{O}_L(a_{n-1})$$

with $a_1 \geq a_2 \geq \dots \geq a_{n-1}$ and $a_1 - a_{n-1} \geq 2$, then $A = R/(f_1, \dots, f_n)$ has not WLP.

Proof. Let ℓ be the general linear form defining a generic line $L \subset \mathbb{P}^2$. To prove the first part, according to Proposition 2.1, we have to show that the multiplication map μ_ℓ is either surjective or injective for every $m \in \mathbb{Z}$. So we consider the long exact sequence in cohomology mentioned above. Firstly, we assume $m < -a - 2$. But then $H^0(L, \mathcal{S}_L(m+1)) = \bigoplus_{i=1}^s H^0(L, \mathcal{O}_L(m+a+1)) \oplus \bigoplus_{i=s+1}^{n-1} H^0(L, \mathcal{O}_L(m+a)) = 0$ and hence μ_ℓ is injective. Now let $m \geq -a - 2$. Then Serre duality yields $H^1(L, \mathcal{S}_L(m+1)) \cong H^0(L, \mathcal{S}_L^*(-m-3))^*$. Since the dual bundle $\mathcal{S}^*(-m-3)$ splits on L as $\bigoplus_{i=1}^s \mathcal{O}_L(-a-m-4) \oplus \bigoplus_{i=s+1}^{n-1} \mathcal{O}_L(-a-m-3)$, it has no non-trivial sections on L . Hence the map μ_ℓ is onto.

For the proof of part two we observe that for the degrees $\deg(\mathcal{S}(-a_1)) < 0$ and $\deg(\mathcal{S}(-a_{n-1})) > 0$ holds. Since $a_1 - a_{n-1} \geq 2$, we can find an $m \in \mathbb{Z}$ with $-a_{n-1} - 2 \geq m+1 \geq -a_1$ and such that $\deg(\mathcal{S}(m+1)) < 0$ and $\deg(\mathcal{S}(m+3)) > 0$. For this m we have $H^0(\mathbb{P}^2, \mathcal{S}(m+1)) = 0$, since the syzygy bundle $\mathcal{S}(m+1)$ is semistable on \mathbb{P}^2 . Because of $a_1 + m + 1 \geq 0$ we also have $H^0(L, \mathcal{S}_L(m+1)) = \bigoplus_{i=1}^{n-1} H^0(L, \mathcal{O}_L(a_i + m + 1)) \neq 0$. Hence the map $H^0(\mathbb{P}^2, \mathcal{S}(m+1)) \rightarrow H^0(L, \mathcal{S}_L(m+1))$ is not surjective and thus μ_ℓ is not injective in degree m . Now we show that μ_ℓ is not surjective either in this particular degree m . Serre duality yields $H^1(L, \mathcal{S}_L(m+1)) \cong H^0(L, \mathcal{S}_L^*(-m-3))^*$ and $H^2(\mathbb{P}^2, \mathcal{S}(m)) \cong H^0(\mathbb{P}^2, \mathcal{S}^*(-m-3))^*$. Since $\mathcal{S}^*(-m-3)$ is semistable and $\deg(\mathcal{S}^*(-m-3)) < 0$ we have $H^0(\mathbb{P}^2, \mathcal{S}^*(-m-3)) = 0$. Now we conclude as above that $H^0(L, \mathcal{S}_L^*(-m-3)) = \bigoplus_{i=1}^{n-1} H^0(L, \mathcal{O}_L(-a_i - m - 3)) \neq 0$ because $-a_{n-1} - m - 3 \geq 0$ is equivalent to $m+1 \leq -a_{n-1} - 2$. So μ_ℓ is not surjective. \square

Remark 2.3. Since we have not used any particular properties of syzygy bundles, Theorem 2.2 can be generalized for arbitrary vector bundles on \mathbb{P}^2 if one translates the Weak Lefschetz property for a vector bundle \mathcal{E} into the property that the multiplication map $H^1(\mathbb{P}^2, \mathcal{E}(m)) \rightarrow H^1(\mathbb{P}^2, \mathcal{E}(m+1))$ induced by a general linear form has maximal rank.

We can now prove [4, Theorem 2.3] easily.

Corollary 2.4. *Every Artinian complete intersection in $K[X, Y, Z]$ has the Weak Lefschetz property.*

Proof. Let $f_1, f_2, f_3 \in R$ be an Artinian complete intersection and let $\ell \in R_1$ be a generic linear form. Firstly, we consider the case that the corresponding

syzygy bundle $\mathcal{S} := \text{Syz}(f_1, f_2, f_3)$ is semistable. Since \mathcal{S} is a 2-bundle, its restriction to the generic line L defined by ℓ splits by the Theorem of Grauert-Müllich as $\mathcal{O}_L(a_1) \oplus \mathcal{O}_L(a_2)$ with $a_1 \geq a_2$ and $0 \leq a_1 - a_2 \leq 1$. Hence by Theorem 2.2(1) the algebra $R/(f_1, f_2, f_3)$ has WLP.

Now suppose \mathcal{S} is not semistable. Since we can pass over to the reflexive hull, the Harder Narasimhan filtration (cf. [7, Definition 1.3.2]) of \mathcal{S} looks like $0 \subset \mathcal{O}_{\mathbb{P}^2}(a) \subset \mathcal{S}$ with $a \in \mathbb{Z}$ (cf. [9, Lemma 1.1.10]). The quotient $\mathcal{F} := \mathcal{S}/\mathcal{O}_{\mathbb{P}^2}(a)$ is a torsion-free sheaf, which is outside codimension 2 isomorphic to its bidual \mathcal{F}^{**} . This bidual is reflexive (cf. [10, Lemma 24.2]), hence locally free on \mathbb{P}^2 (cf. [9, Lemma 1.1.10]), i.e. $\mathcal{F}^{**} \cong \mathcal{O}_{\mathbb{P}^2}(b)$, for $b \in \mathbb{Z}$ and $a > b$. Hence we have $\mathcal{S}|_L \cong \mathcal{O}_L(a) \oplus \mathcal{O}_L(b)$ for a generic line L . If $m < -a - 1$ then $H^0(L, \mathcal{S}_L(m+1)) = 0$ and the multiplication map μ_ℓ is injective. Now, assume that $m \geq -a - 1$. We apply Serre duality and get

$$\begin{aligned} H^1(L, \mathcal{S}_L(m+1)) &= H^0(L, \mathcal{S}_L^*(-m-3))^* \\ &= H^0(L, \mathcal{O}_L(-a-m-3))^* \oplus H^0(L, \mathcal{O}_L(-b-m-3))^* \\ &= H^0(L, \mathcal{O}_L(-b-m-3))^* = H^1(L, \mathcal{O}_L(b+m+1)). \end{aligned}$$

The sheaf morphism $\mathcal{S} \rightarrow \mathcal{F} \rightarrow \mathcal{F}^{**} \cong \mathcal{O}_{\mathbb{P}^2}(b)$ induces a map $H^2(\mathbb{P}^2, \mathcal{S}) \rightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b))$. Hence we have by the functoriality of the connecting homomorphism (cf. [5, Theorem III.1.1.A(d)]) a commutative diagram

$$\begin{array}{ccc} H^1(L, \mathcal{S}_L(m+1)) & \xrightarrow{\delta} & H^2(\mathbb{P}^2, \mathcal{S}(m)) \\ \cong \downarrow & & \downarrow \\ H^1(L, \mathcal{O}_L(b+m+1)) & \xrightarrow{\delta} & H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(b+m)), \end{array}$$

where the bottom map is injective by its explicit description. Hence the map $H^1(L, \mathcal{S}_L(m+1)) \xrightarrow{\delta} H^2(\mathbb{P}^2, \mathcal{S}(m))$ is injective as well. \square

Corollary 2.5. *Let f_1, \dots, f_n , $n \geq 3$, be generic forms in $R = K[X, Y, Z]$ such that their syzygy bundle is semistable. Then $\text{Syz}(f_1, \dots, f_n)$ splits on a generic line L as*

$$\text{Syz}(f_1, \dots, f_n)|_L \cong \bigoplus_{i=1}^s \mathcal{O}_L(a+1) \oplus \bigoplus_{i=s+1}^{n-1} \mathcal{O}_L(a).$$

In particular, if f_1, \dots, f_n are generic polynomials of constant degree d and $3 \leq n \leq 3d$ then the corresponding syzygy bundle has this splitting type.

Proof. Since Anick proved in [1, Corollary 4.14] that every ideal of generic forms f_1, \dots, f_n , $n \geq 3$, in $K[X, Y, Z]$ has the Weak Lefschetz property (in fact Anick proved a much stronger result), this follows immediately from Theorem 2.2(2). The supplement follows from [6, Theorem A.1]. \square

3. ALMOST COMPLETE INTERSECTIONS

In [8, Paragraph after Question 4.2], Migliore and Miró-Roig ask whether *every* almost complete intersection (i.e. four ideal generators) in $K[X, Y, Z]$ has the Weak Lefschetz property. The following easy example gives via Theorem 2.2 a negative answer to this question.

Example 3.1. We consider the monomial almost complete intersection generated by X^3, Y^3, Z^3, XYZ in $K[X, Y, Z]$. The corresponding syzygy bundle $\text{Syz}(X^3, Y^3, Z^3, XYZ)$ is semistable by [2, Corollary 3.6]. We compute its restriction to a line L given by $Z = uX + vY$ with arbitrary coefficients $u, v \in K$. We have

$$Z^3|_L = u^3X^3 + v^3Y^3 + 3uv(uX^2Y + vXY^2) \quad \text{and} \quad (XYZ)|_L = uX^2Y + vXY^2.$$

For $u, v \neq 0$, and in particular for generic u, v , this gives immediately the non-trivial syzygy

$$u^3X^3 + v^3Y^3 + 3uv(uX^2Y + vXY^2) - u^3X^3 - v^3Y^3 - 3uv(uX^2Y + vXY^2),$$

which yields a non-trivial global section in $\text{Syz}(X^3, Y^3, Z^3, XYZ)(3)|_L$. Since $\text{Syz}(X^3, Y^3, Z^3, XYZ)(3)$ has degree $-3 < 0$, this section does not come from \mathbb{P}^2 . Moreover,

$$\text{Syz}(X^3, Y^3, Z^3, XYZ)(4)|_L \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L \oplus \mathcal{O}_L(-1),$$

hence by Theorem 2.2 the Artinian algebra

$$A = K[X, Y, Z]/(X^3, Y^3, Z^3, XYZ)$$

has not the Weak Lefschetz property. Indeed, the map $A_2 \rightarrow A_3$ given by a generic linear form is neither injective nor surjective since $v^2X^2 + u^2Y^2 + Z^2 - uvXY - vXZ - uYZ$ is in the kernel for every generic linear form $\ell = uX + vY + Z$ and $\dim_K A_2 = 6 = \dim_K A_3$.

The following reasoning shows that this is the only counterexample in degree 3 containing the monomials X^3, Y^3, Z^3 . So we consider the monomials X^3, Y^3, Z^3 and a forth homogeneous polynomial $f = \sum_{|\nu|=3} a_\nu X^\nu$ of degree 3. If we use again $Z = uX + vY$ to restrict the corresponding syzygy bundle to a generic line, we have to compute the coefficients $c_{(1,2)}$ and $c_{(2,1)}$ of the monomials XY^2 and X^2Y in f restricted to L . These are

$$c_{(1,2)} = a_{(1,2,0)} + a_{(0,2,1)}u + 2a_{(0,1,2)}uv + a_{(1,1,1)}v + a_{(1,0,2)}v^2 + 3a_{(0,0,3)}uv^2$$

and

$$c_{(2,1)} = a_{(2,1,0)} + a_{(0,1,2)}u^2 + a_{(2,0,1)}v + a_{(1,1,1)}u + 2a_{(1,0,2)}uv + 3a_{(0,0,3)}u^2v.$$

The algebra $A = K[X, Y, Z]/(X^3, Y^3, Z^3, f)$ has not the Weak Lefschetz property if and only if there exists a non-trivial global section of $S|_L(3)$, and this is true if $c_{(2,1)}X^2Y + c_{(1,2)}XY^2$ is a multiple of $uX^2Y + vXY^2$, i.e. $c_{(2,1)}X^2Y + c_{(1,2)}XY^2 = t(uX^2Y + vXY^2)$ for some $t \in K$. This means

$c_{(2,1)} = tu$ and $c_{(1,2)} = tv$ and gives the condition $vc_{(2,1)} - uc_{(1,2)} = 0$. We have $vc_{(2,1)} - uc_{(1,2)} =$

$$va_{(2,1,0)} - ua_{(1,2,0)} + v^2a_{(2,0,1)} - u^2a_{(0,2,1)} - u^2va_{(0,1,2)} + uv^2a_{(1,0,2)}.$$

If we consider the right hand side of this equation as a polynomial in $K[u, v]$ and assume that at least one of the coefficients is not zero, then there exists also values in K where this polynomial does not vanish (K is an infinite field). Hence we see that if $f \notin (X^3, Y^3, Z^3, XYZ)$ this condition can not hold for all u, v and therefore the algebra A has the Weak Lefschetz property.

Remark 3.2. In [8, Question 4.2], Migliore and Miró-Roig asked: “For any integer $n \geq 3$, find the maximum number $A(n)$ (if it exists) such that *every* Artinian ideal $I \subset k[x_1, \dots, x_n]$ with $\mu(I) \leq A(n)$ has the Weak Lefschetz property (where $\mu(I)$ is the minimum number of generators of I).” They show in [8, Example 4.2] that the Artinian ideal $I := (X^2, Y^2, Z^2, XY, XZ)$ has not the Weak Lefschetz property, so $A(3) \leq 4$. Since by Corollary 2.4 every complete intersection in $K[X, Y, Z]$ has the Weak Lefschetz property, Example 3.1 proves that $A(3) = 3$. Furthermore, Example 3.1 shows also that

$$XYZ \in \ker[(R/(X^3, Y^3, Z^3))_3 \longrightarrow (R/(\ell, X^3, Y^3, Z^3))_3]$$

for all linear forms $\ell \in R_1$. Hence, the last hypothesis in [8, Proposition 5.5] does not always hold.

For an almost complete intersection in $K[X, Y, Z]$ we can prove the following theorem.

Theorem 3.3. *Let f_1, f_2, f_3, f_4 be R_+ -primary homogeneous polynomials in $R = K[X, Y, Z]$ such that their syzygy bundle is not semistable. Then the algebra $R/(f_1, f_2, f_3, f_4)$ has the Weak Lefschetz property.*

Proof. We consider the Harder-Narasimhan filtration of the vector bundle $\mathcal{S} := \text{Syz}(f_1, f_2, f_3, f_4)$. As in the proof of Corollary 2.4 we can pass over to the reflexive hull and since reflexive sheaves on \mathbb{P}^2 are locally free (cf. [9, Lemma 1.1.10]), we can assume that all subsheaves in the HN-filtration are vector bundles. Further we fix a generic line $L = V_+(\ell)$, where $\ell \in R_1$ is a general linear form.

Firstly, we treat the case that the destabilizing subbundle is a line bundle $\mathcal{O}_{\mathbb{P}^2}(a_1)$ for some $a_1 \in \mathbb{Z}$, i.e. we have an exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(a_1) \rightarrow \mathcal{S} \rightarrow \mathcal{E} \rightarrow 0$ with a semistable torsion-free sheaf \mathcal{E} of rank 2. We apply the Theorem of Grauert-Müllich to \mathcal{E}^{**} , which is locally free and outside codimension 2 isomorphic to \mathcal{E} . Therefore we get $\mathcal{E}|_L \cong \mathcal{O}_L(a_2) \oplus \mathcal{O}_L(a_3)$ with $a_2 \geq a_3$ and $0 \leq a_2 - a_3 \leq 1$. Since the slopes in the HN-filtration decrease strictly we have $2a_1 > a_2 + a_3 \geq a_2 + a_2 - 1 = 2a_2 - 1$ and therefore $a_1 \geq a_2 \geq a_3$. In particular $\mathcal{S}|_L \cong \mathcal{O}_L(a_1) \oplus \mathcal{O}_L(a_2) \oplus \mathcal{O}_L(a_3)$. For $m < -a_2 - 1$ we have $H^0(L, \mathcal{O}_L(a_2 + m + 1) \oplus \mathcal{O}_L(a_3 + m + 1)) = 0$ (for $m < -a_1 - 1$ we have

even $H^0(L, \mathcal{S}_L(m+1)) = 0$). Therefore the injectivity of μ_ℓ follows from the diagram

$$\begin{array}{ccc} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a_1 + m + 1)) & \longrightarrow & H^0(L, \mathcal{O}_L(a_1 + m + 1)) \\ \downarrow & & \downarrow \cong \\ H^0(\mathbb{P}^2, \mathcal{S}(m+1)) & \longrightarrow & H^0(L, \mathcal{S}_L(m+1)). \end{array}$$

For $m \geq -a_2 - 1$ we get $H^1(L, \mathcal{S}_L(m+1)) = H^0(L, \mathcal{S}_L^*(-m-3))^* = 0$, since $-a_3 - m - 3 \leq 1 - a_2 - m - 3 = -a_2 - m - 2 < 0$ holds. Hence μ_ℓ is surjective.

Now suppose that \mathcal{S} is destabilized by a semistable vector bundle \mathcal{E} of rank 2. (This case corresponds essentially to [8, Proposition 5.2], where one of the ideal generators has a sufficiently large degree.) Here we apply the Theorem of Grauert-Mülich to \mathcal{E} and get $\mathcal{E}|_L \cong \mathcal{O}_L(a_1) \oplus \mathcal{O}_L(a_2)$ with $a_1 \geq a_2$ and $0 \leq a_1 - a_2 \leq 1$. The quotient $\mathcal{F} := \mathcal{S}/\mathcal{E}$ is outside codimension 2 isomorphic to $\mathcal{O}_{\mathbb{P}^2}(a_3)$ with $a_3 \in \mathbb{Z}$ and since \mathcal{F} is a quotient in the HN-filtration we get again $a_1 \geq a_2 \geq a_3$ and $\mathcal{S}|_L \cong \mathcal{O}_L(a_1) \oplus \mathcal{O}_L(a_2) \oplus \mathcal{O}_L(a_3)$. Hence for $m < -a_1 - 1$ we have $H^0(L, \mathcal{S}_L(m+1)) = 0$ and thus μ_ℓ is injective. We treat the case $m \geq -a_1 - 1$ similar to the analog situation in the proof of Corollary 2.4. By Serre duality we get $H^1(L, \mathcal{S}_L(m+1)) = H^0(L, \mathcal{S}_L^*(-m-3))^* = H^0(L, \mathcal{O}_L(-a_3 - m - 3))^* = H^1(L, \mathcal{O}_L(a_3 + m + 1))$, since $H^0(L, \mathcal{O}_L(-a_1 - m - 3) \oplus \mathcal{O}_L(-a_2 - m - 3)) = 0$. The map $\mathcal{S} \rightarrow \mathcal{F} \rightarrow \mathcal{F}^{**} \cong \mathcal{O}_{\mathbb{P}^2}(a_3)$ induces a map $H^2(\mathbb{P}^2, \mathcal{S}) \rightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a_3))$ between the second cohomology groups. Therefore the injectivity of the map $H^1(L, \mathcal{S}_L(m+1)) \xrightarrow{\delta} H^2(\mathbb{P}^2, \mathcal{S}(m))$ follows from the injectivity of the map $H^1(L, \mathcal{O}_L(a_3 + m + 1)) \xrightarrow{\delta} H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a_3 + m))$. Hence the map μ_ℓ is surjective and the Artinian algebra $R/(f_1, f_2, f_3, f_4)$ has WLP.

Finally we consider a HN-filtration of the form $0 \subset \mathcal{O}_{\mathbb{P}^2}(a_1) \subset \mathcal{E} \subset \mathcal{S}$ with a vector bundle \mathcal{E} of rank 2. Since the quotients of this filtration are torsion-free, we have outside codimension 2 the identifications $\mathcal{E}/\mathcal{O}_{\mathbb{P}^2}(a_1) \cong \mathcal{O}_{\mathbb{P}^2}(a_2)$ and $\mathcal{S}/\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2}(a_3)$ with $a_1 > a_2 > a_3$. Therefore, we have on the generic line L the splitting $\mathcal{S}|_L \cong \mathcal{O}_{\mathbb{P}^2}(a_1) \oplus \mathcal{O}_{\mathbb{P}^2}(a_2) \oplus \mathcal{O}_{\mathbb{P}^2}(a_3)$. For $m < -a_2 - 1$ we have $H^0(L, \mathcal{O}_L(a_2 + m + 1) \oplus \mathcal{O}_L(a_3 + m + 1)) = 0$, i.e. $H^0(L, \mathcal{S}_L(m+1)) = H^0(L, \mathcal{O}_L(a_1))$. Since $0 \rightarrow \mathcal{O}(a_1) \rightarrow \mathcal{S}$ the map $H^0(\mathbb{P}^2, \mathcal{S}(m+1)) \rightarrow H^0(L, \mathcal{S}_L(m+1))$ is onto. Now, let $m \geq -a_2 - 1$. Here we have $H^1(L, \mathcal{S}_L(m+1)) = H^1(L, \mathcal{O}_L(a_3 + m + 1))$ and $\mathcal{S} \rightarrow (\mathcal{S}/\mathcal{E})^{**} \cong \mathcal{O}_{\mathbb{P}^2}(a_3)$. Hence the injectivity of the map $H^1(L, \mathcal{S}_L(m+1)) \xrightarrow{\delta} H^2(\mathbb{P}^2, \mathcal{S}(m))$ follows again from the injectivity of $H^1(L, \mathcal{O}_L(a_3 + m + 1)) \xrightarrow{\delta} H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a_3 + m))$ and we are done. \square

We want to give examples which show that all the three possible Harder-Narasimhan filtrations mentioned in the proof of Theorem 3.3 can appear. It

is even possible to provide monomial examples for all these cases. For degree computations of syzygy bundles see [2, Lemma 2.1].

Examples 3.4. The syzygy bundle for the monomials X^4, Y^4, Z^4, X^3Y has degree -16 , hence its slope equals $-16/3 \approx -5.33$. This vector bundle is not semistable because the subsheaf $\text{Syz}(X^4, X^3Y)$ has degree $-5 > -5.33$. Since the degrees are constant, there are no maps into line bundles which contradict the semistability. Therefore

$$\mathcal{O}_{\mathbb{P}^2}(-5) \cong \text{Syz}(X^4, X^3Y) \subset \text{Syz}(X^4, Y^4, Z^4, X^3Y)$$

constitutes the HN-filtration of the syzygy bundle. The generic splitting type is $\mathcal{O}_L(-5) \oplus \mathcal{O}_L(-5) \oplus \mathcal{O}_L(-6)$.

To give an example for the second type of HN-filtration consider the monomials $X^4, Y^4, Z^4, X^3Y^3Z^3$. The corresponding syzygy bundle has the slope $-21/3 = -7$ and it is destabilized by the semistable subsheaf $\text{Syz}(X^4, Y^4, Z^4)$ of slope $-12/2 = -6$. Hence we have found the HN-filtration of the bundle $\text{Syz}(X^4, Y^4, Z^4, X^3Y^3Z^3)$. The generic splitting type is $\mathcal{O}_L(-6) \oplus \mathcal{O}_L(-6) \oplus (-9)$.

For the third type of HN-filtration we look at the family X^2, Y^4, Z^7, XY . The HN-filtration of $\text{Syz}(X^2, Y^4, Z^7, XY)$ is

$$0 \subset \mathcal{O}_{\mathbb{P}^2}(-3) \cong \text{Syz}(X^2, XY) \subset \text{Syz}(X^2, XY, Y^4) \subset \text{Syz}(X^2, Y^4, Z^7, XY),$$

since the quotients have rank one and degrees $-3 > -5 > -7$. Accordingly, the generic splitting type is $\mathcal{O}_L(-3) \oplus \mathcal{O}_L(-5) \oplus \mathcal{O}_L(-7)$.

By combining Theorem 3.3 with Theorem 2.2 we see that almost complete intersections in $K[X, Y, Z]$ are now well understood with respect to the Weak Lefschetz property.

REFERENCES

- [1] D. Anick, *Thin algebras of embedding dimension three*, J. Algebra **100** (1986), 235–259.
- [2] H. Brenner, *Looking out for stable syzygy bundles*, ArXiv (2003).
- [3] R. Fröberg, *An inequality for Hilbert series of graded algebras*, Math. Scand. **56** (1985), 117–144.
- [4] T. Harima, J. C. Migliore, U. Nagel, and J. Watanabe, *The weak and strong Lefschetz properties for Artinian k -algebras*, J. Algebra **262** (2003), 99–126.
- [5] R. Hartshorne, *Algebraic geometry*, Springer, 1977.
- [6] G. Hein, *Semistability of the general syzygy bundle*, Appendix to H. Brenner, Looking out for stable syzygy bundles, ArXiv (2003).
- [7] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Viehweg, 1997.
- [8] J. Migliore and R. M. Miró-Roig, *Ideals of general forms and the ubiquity of the Weak Lefschetz property*, J. Pure App. Alg. (2003), 79–107.
- [9] C. Okonek, M. Schneider, and H. Spindler, *Vector bundles on complex projective spaces*, Birkhäuser, 1980.
- [10] G. Scheja and U. Storch, *Lokale Verzweigungstheorie*, vol. 5, Schriftenreihe des mathematischen Institutes der Universität Freiburg, 1974.

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF SHEFFIELD, HICKS BUILDING,
HOUNSFIELD ROAD, SHEFFIELD S3 7RH, UNITED KINGDOM
E-mail address: `H.Brenner@sheffield.ac.uk` and `A.Kaid@sheffield.ac.uk`